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Verification of a non-hydrostatic dynamical core using horizontally spectral element vertically finite difference method: 2D Aspects

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14. ABSTRACT

The non-hydrostatic (NH) compressible Euler equations of dry atmosphere are solved in a simplified two dimensional (2D) slice (X-Z) framework employing a spectral element method (SEM) for the horizontal discretization and a finite difference method (FDM) for the vertical discretization. The SEM uses high-order nodal basis functions associated with Lagrange polynomials based on Gauss-Lobatto-Legendre (GLL) quadrature points. The FDM employs a third-order upwind biased scheme for the vertical flux terms and a centered finite difference scheme for the vertical derivative terms and quadrature. The Euler equations used here are in a flux form based on the hydrostatic pressure vertical coordinate, which are the same as those used in the Weather Research and Forecasting (WRF) model, but a hybrid sigma-pressure vertical coordinate is implemented in this model. We verified the model by conducting widely used standard benchmark tests: the inertia-gravity wave, rising thermal bubble, density current wave, and linear hydrostatic mountain wave. The numerical results demonstrate that the horizontally spectral element vertically finite difference model is accurate and robust. By using the 2D slice model, we effectively show that the combined spatial discretization method of the spectral element and finite difference method in the horizontal and vertical directions, respectively, offers a viable method for the development of a NH dynamical core. The present core provides a practical framework for further development of three-dimensional (3D) non-hydrostatic compressible atmospheric models.

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Abstract

The non-hydrostatic (NH) compressible Euler equations of dry atmosphere are solved in a simplified two dimensional (2D) slice (X-Z) framework employing a spectral element method (SEM) for the horizontal discretization and a finite difference method (FDM) for the vertical discretization. The SEM uses high-order nodal basis functions associated with Lagrange polynomials based on Gauss-Lobatto-Legendre (GLL) quadrature points. The FDM employs a third-order upwind biased scheme for the vertical flux terms and a centered finite difference scheme for the vertical derivative terms and quadrature. The Euler equations used here are in a flux form based on the hydrostatic pressure vertical coordinate, which are the same as those used in the Weather Research and Forecasting (WRF) model, but a hybrid sigma-pressure vertical coordinate is implemented in this model. We verified the model by conducting widely used standard benchmark tests: the inertia-gravity wave, rising thermal bubble, density current wave, and linear hydrostatic mountain wave. The numerical results demonstrate that the horizontally spectral element vertically finite difference model is accurate and robust. By using the 2D slice model, we effectively show that the combined spatial discretization method of the spectral element and finite difference method in the horizontal and vertical directions, respectively, offers a viable method for the development of a NH dynamical core. The present core provides a practical framework for further development of three-dimensional (3D) non-hydrostatic compressible atmospheric models.

1. Introduction

There is a growing interest in developing highly scalable dynamical cores using numerical algorithms under petascale computers with many cores (with the goal of exascale computing just around the corner). The spectral element method (SEM) has been known as one of the most promising methods with high efficiency and accuracy (Taylor et al. 1997; Giraldo 2001; Thomas and Loft 2002). SEM is local in nature because of having a large on-processor operation count (Kelly and Graldo, 2012). The SEM achieves this high level of scalability by decomposing the physical domain into smaller pieces with a small communication stencil. Also SEM has been shown to be very attractive in achieving high-order accuracy and geometrical flexibility on the sphere (Taylor et al. 1997; Giraldo 2001; Giraldo et al. 2004).

To date, the SEM has been successfully implemented in atmospheric modeling such as in the Community Atmosphere Model – spectral element dynamical core (CAM-SE) (Thomas and Loft 2005) and the scalable spectral element Eulerian atmospheric model (SEE-AM) (Giraldo and Rosmond, 2004). These models consider the primitive hydrostatic equations on global grid meshes such as a cubed-sphere tiled with quadrilateral elements using SEM in the horizontal discretization and the finite difference method (FDM) in the vertical. The robustness of the SEM has been illustrated through three-dimensional dry dynamical test cases (Thomas and Loft 2005; Giraldo and Rosmond 2004; Giraldo 2005; Taylor et al. 2007; Lauritzen et al. 2010).

The ultimate objective of our study is to build a 3D non-hydrostatic (NH) model based on the compressible Navier-Stokes equations using the combined horizontally SEM and vertically FDM. Since testing a 3D NH model requires a huge amount of computing resources, studying the feasibility of our approach in 2D is an attractive alternatively to the

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69 development of a fully 3D model. This is the case because a 2D slice model effectively can

70 test the practical issues resulting from the vertical discretization and time integration, prior to

71 the construction of a full 3D model. Although we could also discretize the vertical direction

72 with SEM (as is proposed in Kelly and Giraldo 2012 and Giraldo et al. 2013), we choose to

73 use a conservative flux-form finite-difference method for discretization in the vertical

74 direction, which provides an easy way for coupling the dynamics and existing physics

75 packages.

76   We have developed a dry 2D NH compressible Euler model based on SEM along the x-

77 direction and FDM along the z-direction for this purpose. Hereafter, this is simply referred to

78 as the 2DNH model. We adopt the governing equation formulation proposed by Skamarock

79 and Klemp (2008) (hereafter, SK08) which is used in the Weather Research and Forecasting

80 (WRF) Model. The Euler equations are in flux form based on the hydrostatic pressure vertical

81 coordinate. In SK08, the terrain-following sigma-pressure coordinate is used, but here we

82 employ a hybrid sigma-pressure vertical coordinate. Park et al. (2013) (hereafter, PK13)

83 provides a clue for the equation set in the hybrid sigma-pressure in their appendix, in which

84 the hybrid sigma-pressure coordinate is applied to the hydrostatic primitive equations and can

85 be modified exactly to the sigma-pressure coordinate at the level of the actual coding

86 implementation. Also, we built the 2DNH model using a time-split third-order Runge-Kutta

87 (RK3) for the time discretization, which has been shown to work effectively in the WRF

88 model. We keep the temporal discretization of the model as similar as possible to the WRF

89 model in order to more directly the discern the differences related to the discrete spatial

90 operators between the two models. This provides robust tools for development and

91 verification of the 2DNH model.

92   In this paper, we show the feasibility of the 2DNH model by conducting conventional

93 benchmark test cases as well as focusing on the description of the numerical scheme for the

94   spatial discretization. We verify the 2DNH by analyzing four test cases: the inertia-gravity

95   wave, rising thermal bubble, density current wave, and linear hydrostatic mountain wave.

96   The organization of this paper is as follows. The next section describes the governing

97   equations with a definition of the prognostic and diagnostic variables used in our model, in

98   which we present essential changes from SK08. Section 3 contains the description of the

99   temporal and spatial discretization including the spectral element formulation. In Sec. 4, we

100   present the results of the 2DNH model using all four test cases. Finally in Sec. 5 we

101   summarize the paper and propose future directions.

102

2. Governing equations

104   We adopt the governing equation formulation of SK08. Here we implement the hybrid

105   sigma-pressure coordinate reported by PK13 which considered only the hydrostatic primitive

106   equation. The hybrid sigma pressure coordinate is defined with $\eta \in \left[0,1\right]$ as

$$p_d = B(\eta)\left(p_s - p_t\right) + \left[\eta - B(\eta)\right]\left(p_0 - p_t\right) + p_t \tag{1}$$

108   where p_d is the hydrostatic pressure of dry air, $B(\eta)$ is the relative weighting of the

109   terrain-following coordinate versus the normalized pressure coordinate, p_s, p_t, and p_0 are

110   the hydrostatic surface pressure of dry air, the top level pressure, and a reference sea level

111   pressure, respectively. A more detailed description of the hybrid sigma pressure coordinate

112   can be found in the Appendix of PK13. The definition of the flux variables are

$$\left(\vec{V}_H, W, \Omega, \Theta\right) = \mu_d \times \left(\vec{v}_H, w, \dot{\eta}, \theta\right) \tag{2}$$

114   where $\vec{v}_H = \left(u,v\right)$ and w are the velocities in the horizontal and vertical directions,

115   respectively, $\dot{\eta} \equiv \dfrac{\partial \eta}{\partial t}$ is the η-coordinate (contravariant) vertical velocity, θ is the

116   potential temperature, and μ_d is the mass of the dry air in the layers defined as

117
$$\mu_d(x,y,\eta,t) = \frac{\partial p_d}{\partial \eta} = \frac{\partial B(\eta)}{\partial \eta}(p_s - p_t) + \left[1 - \frac{\partial B(\eta)}{\partial \eta}\right](p_0 - p_t). \tag{3}$$

118   The flux-form Euler equations for dry atmosphere are expressed as

119
$$\frac{\partial \vec{V}_H}{\partial t} = -\mu_d\left(\nabla_\eta \phi' + \alpha_d \nabla_\eta p' + \alpha_d' \nabla_\eta \bar{p}\right) - \nabla_\eta \phi\left(\frac{\partial p'}{\partial \eta} - \mu_d'\right) - \nabla_\eta \cdot \left(\vec{V}_H \otimes \vec{v}_H\right) - \frac{\partial(\Omega \vec{v}_H)}{\partial \eta} + F_{\vec{V}_H}, \tag{4}$$

120
$$\frac{\partial W}{\partial t} = g\left[\frac{\partial p'}{\partial \eta} - \mu_d'\right] - \nabla_\eta \cdot \left(\vec{V}_H w\right) - \frac{\partial(\Omega w)}{\partial \eta} + F_W, \tag{5}$$

121
$$\frac{\partial \mu_d'}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial p_d'}{\partial \eta}\right) = \frac{\partial B(\eta)}{\partial \eta}\frac{\partial p_s'}{\partial t} = -\nabla_\eta \cdot \vec{V}_H - \frac{\partial \Omega}{\partial \eta}, \tag{6}$$

122
$$\frac{\partial \phi'}{\partial t} = -\frac{1}{\mu_d}\left[\vec{V}_H \cdot \nabla_\eta \phi + \Omega\frac{\partial \phi}{\partial \eta} - gW\right], \tag{7}$$

123
$$\frac{\partial \Theta}{\partial t} = -\nabla_\eta \cdot \left(\vec{V}_H \theta\right) - \frac{\partial(\Omega \theta)}{\partial \eta}, \tag{8}$$

124   where ϕ is the geopotential, α_d is the inverse density for dry air, and $F_{V_H}^r$ and F_W

125   represent forcing terms of the Coriolis and curvature which we ignore for simplicity. In Eqs.

126   (4)-(8), the governing equations are described with perturbation variables such as

127     $p = \bar{p}(\bar{z}) + p'$, $\phi = \bar{\phi}(\bar{z}) + \phi'$, $\alpha_d = \bar{\alpha}_d(\bar{z}) + \alpha_d'$, and $p_s = \bar{p}_s(x,y) + p_s'$ where the

128   variables denoted by overbars are reference state variables that satisfy hydrostatic balance.

129   For completeness, the diagnostic relation for Ω is given by integrating Eq. (6)

130   vertically from the surface ($\eta = 1$) to the material surface as

131
$$\Omega = -\int_1^\eta \left(\frac{\partial B(\eta)}{\partial \eta}\frac{\partial p_s'}{\partial t} + \nabla_\eta \cdot \vec{V}_H\right)d\eta, \tag{9}$$

132   where $\dfrac{\partial p_s'}{\partial t}$ is obtained by integrating vertically Eq. (6) for the surface ($\eta = 1$) to the top

133 ($\eta = 0$) using a no-flux boundary condition such as $\left.\Omega\right|_{\eta=0 \text{ or } 1} = 0$ and the specification of

134 the vertical coordinate such as $B(\eta = 1) = 1$ and $B(\eta = 0) = 0$ as

$$\frac{\partial p_s'}{\partial t} = -\int_{\eta=0}^{\eta=1} \left(\nabla \cdot \vec{V}_H\right) d\eta . \tag{10}$$

135

136 The above equation allows p_s' to be evolved forward in time where we then compute μ_d'

137 directly from Eq. (5). The diagnostic relation for the dry inverse density is given as

$$\frac{\partial \phi'}{\partial \eta} = -\bar{\mu}_d \alpha_d' - \alpha_d \mu_d' , \tag{11}$$

138

139 and the full pressure for dry atmosphere is

$$p = p_0 \left(\frac{R_d \theta}{p_0 \alpha_d}\right)^{c_p/c_v} . \tag{12}$$

140

141 This concludes the description of the governing equations used in our model; in the next

142 section we describe the discretization of the continuous form of the governing equations that

143 are used in our model.

3. Discretization

145 a. Spatial discretization

146 1) Horizontal direction

147 For a given η level, we discretize the horizontal operators using the SEM. Therefore in

148 2D (X-Z) slice framework we focus on the SEM discrete gradient operator for 1D (x). In

149 SEM, we approximate the solution in non-overlapping elements Ω^e as

$$q(x,t) = \sum_{k=1}^{N+1} \psi_k(x) q_N(x_k, t) , \tag{13}$$

150

151 where x_k represents $N+1$ grid points that correspond to the Gauss-Lobatto-Legendre

152 (GLL) points and $\psi_k(x)$ are the N th-order Lagrange polynomials based on the GLL points.

153   It is noteworthy that the ψ_k have the cardinal property, i.e., they can be represented as

154   Kronecker delta functions where ψ_k are zero at all nodal points except x_k (but are

155   allowed to oscillate between nodal points).

156   The GLL points ξ_k in a reference coordinate system $\xi \in \left[-1,+1\right]$ and the associated

157   quadrature weights $\omega(\xi_k)$,

$$\omega(\xi_k) = \frac{2}{N\left(N+1\right)}\left[\frac{1}{P_N(\xi_k)}\right]^2, \tag{14}$$

159   are introduced for the Gaussian quadrature:

$$\int_{\Omega^e} q \ d\,\Omega^e = \int_{-1}^{+1} q(\xi)\left|J\left(\xi\right)\right| d\,\xi \approx \sum_{i=0}^{N} \omega(\xi_i)q(\xi_i)\left|J\left(\xi_i\right)\right|, \tag{15}$$

161   where $P_N(\xi)$ are the N th-order Legendre polynomials, $J = \dfrac{\partial x}{\partial \xi}$ is the transformation

162   Jacobian, and Ω^e represents the non-overlapping elements.

163   We now introduce the polynomial expansions into our governing equations in the form

164   of

$$\frac{\partial q}{\partial t} = -F(q), \tag{16}$$

165   multiply by the basis function as a test function, and integrate to yield a system of ordinary

166   differential equations as such

$$\sum_{n=1}^{N+1} M_{nk}^e \frac{dq_k}{dt} = -\int_{\Psi_e} F\left(\sum_{n=1}^{N+1} \psi_n(\xi)q_n\right)\psi_k \ d\xi, \tag{17}$$

168   where $k = 1,2,\text{L}\,,N+1,$ M_{nk}^e is the element based mass matrix given as

$$M_{nk}^e = \int_{\Psi_e} \psi_n \psi_k \ d\xi = \omega_n\left|J_n\right|\delta_{nk}, \tag{18}$$

170   and the right-hand side of Eq. (18) is evaluated using Gaussian quadrature of Eq. (16). It is

171   noted that using GLL points for both interpolation and integration results in a diagonal mass

172 matrix M_{nk}^{e}, which means that the inversion of the mass matrix is trivial.

173 The horizontal derivatives included in the right-hand side of Eq. (17) are evaluated using

174 the analytic derivatives of the basis functions as follows

$$\frac{\partial q}{\partial x} = \frac{\partial q}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial \xi}\left[\sum_{k=1}^{N+1}\psi_k(\xi)q_k\right]\frac{\partial \xi}{\partial x} = \left[\sum_{k=1}^{N+1}\frac{\partial \psi_k}{\partial \xi}q_k\right]\frac{1}{|J|}. \tag{19}$$

176 Note that the non-differential operations, such as cross products, are computed directly at grid

177 points since we use nodal basis functions associated with Lagrange polynomials based on the

178 GLL points. In order to satisfy the equations globally, we use the direct stiffness summation

179 (DSS) operation. For a more detailed description of the SEM, see Giraldo and Rosmond

180 (2004), Giraldo and Restelli (2008), and Kelly and Giraldo (2012).

181

182 2) Vertical direction

183 Using a Lorenz staggering, the variables $\vec{V}_H$, Θ, μ, α, p are at layer midpoints

184 denoted by $k = 1,2,\mathrm{K},K$ where K is the total number of layers, and the variables W,

185 Ω, ϕ live at layer interfaces defined by $k+\frac{1}{2}$, $k = 0,1,\mathrm{K},K$, so that $\eta_{K+1/2} = \eta_{top}$ and

186 $\eta_{1/2} = \eta_{Bottom} = 1$. Fig. 1 describes the grid points and the allocation of the variables. Here, we

187 evaluate the vertical advection terms ($\dfrac{\partial(\Omega\vec{V}_H)}{\partial\eta}$, $\dfrac{\partial(\Omega w)}{\partial\eta}$, and $\dfrac{\partial(\Omega\theta)}{\partial\eta}$) and vertical

188 derivative terms ($\dfrac{\partial p'}{\partial\eta}$, and $\dfrac{\partial\phi}{\partial\eta}$). The former is discretized using the third-order upwind

189 biased discretization in Hundsdorfer et al. (1995) which is given as

$$\left.\frac{\partial f}{\partial\eta}\right|_k = \frac{f_{k-2} - 8f_{k-1} + 8f_{k+1} - f_{k+2}}{12\Delta\eta} + \mathrm{sign}(\Omega)\frac{f_{k-2} - 4f_{k-1} + 6f_k - 4f_{k+1} + f_{k+2}}{12\Delta\eta}, \tag{20}$$

191 where f corresponds to the flux such as $\Omega\vec{V}_H$, and $\Delta\eta = \eta_{k+1/2} - \eta_{k-1/2}$ is the thickness of

192 the layer. The latter is discretized by the centered finite difference. Likewise the vertical

193 discretization quadrature rules for the calculations of Eqs. (9) and (10) follow the finite

194 difference discretization naturally.

195

196   b. Temporal discretization

197   For integrating the equations, we use the time-split RK3 integration technique following

198 the strategy of SK08, in which low-frequency modes due to advective forcings are explicitly

199 advanced using a large time step of the RK3 scheme, but high-frequency modes are

200 integrated over smaller time steps using an explicit forward-backward time integration

201 scheme for the horizontally propagating acoustic/gravity waves and a fully implicit scheme

202 for vertically propagating acoustic waves and buoyancy oscillations (Klemp et al. 2007) .

203 This technique has been shown to work effectively within numerous nonhydrostatic models

204 including the WRF model (Skamarock et al. 2008), the Model for Prediction Across Scales

205 (MPAS) (Skamarock et al. 2012), and the Nonhydrostatic Icosahedral Atmospheric Model

206 (NICAM) (Satoh et al. 2008).

207   It is noted that in the procedure of the time-split RK3 integration, the difference between

208 the approach used in this paper and SK08 comes from the vertical coordinate. Since we use

209 the hybrid sigma-pressure coordinate, the equation for p'_s (Eq. (6)) should be first stepped

210 forward in time using the forward-backward differencing on the small time steps, then μ'_d

211 can be computed directly from the specification of the vertical coordinate in Eq. (9) and Ω

212 is obtained from the vertical integration.

213

214 **4. Test cases**

215   We validate the 2DNH model on four test cases of the linear hydrostatic mountain wave,

216 density current, inertia-gravity wave, and rising thermal bubble experiments. All cases but the

217 mountain wave experiment do not have analytic solutions. Therefore, for the mountain wave

218 experiment, numerical results of the 2DNH model are compared to analytic solutions (Durran

219 and Klemp 1983), and for the other experiments, we compare our results to the results of

220 other published papers.

221   It should be mentioned that the horizontal SEM formulation is able to utilize arbitrary

222 order polynomials per element to represent the discrete spatial operators, but in this paper all

223 the results presented use either 5th or 8th order polynomials. The averaged horizontal grid

224 spacing is defined as

$$\Delta \bar{x} = \frac{\sum_{n=1}^{N} \Delta x_n}{N} \tag{21}$$

225

226 where Δx_n is the internal grid spacing within the element which is regularly spaced in the

227 domain and N is the number of the interval associated with irregularly spaced GLL

228 quadrature points which is equivalent to the order of the basis polynomials. The average

229 vertical grid spacing is defined in the same way of Eq. (24). Below, we use this convention to

230 define the grid resolution.

231

232   a. Linear hydrostatic mountain wave test

233   In order to verify the 2DNH's feasibility to treat surface elevations associated with the

234 vertical terrain-following coordinate, we simulate the linear hydrostatic mountain wave test

235 introduced by Durran and Klemp (1983) (hereafter, DK83) in which the analytic steady-state

236 solution is provided by using a single-peaked mountain with uniform zonal wind. To compare

237 our results with the analytic and numerical solution shown in DK83, the 2DNH is initialized

238 using the same initial conditions and mountain profile in DK83 and we analyze our results

239   using the same metrics of DK83.

240   The mountain profile is given by

$$h(x) = \frac{h_m}{1 + \left(\dfrac{x - x_c}{a_m}\right)^2} \tag{22}$$

242   where the half-length of the mountain a_m is 10 km, the height h_m is 1 m, and the

243   prescribed center of the profile is 0 km. The Initial temperature is $T_0 = 250$ K for an

244   isothermal atmosphere with the uniform zonal wind $\bar{u} = 20$ m/s. In the isothermal case, the

245   Brunt-Väisälä frequency $N^2 = g\dfrac{d(\ln\bar{\theta})}{dz} \approx \dfrac{g^2}{c_p T_0}$ yields the potential temperature given as

$$\bar{\theta} = \theta_0 e^{\frac{g}{c_p T_0} z}, \tag{23}$$

247   which is one of the prognostic variables in our model. The domain is defined as

248    $(x, z) \in [-300, 300] \times [0, 30]$ km^2. The bottom boundary uses a no-flux boundary

249   condition while the lateral and top boundaries use sponge layers. The sponged zone is 10 km

250   deep from the top and 50 km wide from the lateral boundaries. Over the sponge layer zone,

251   the prognostic variables are relaxed to the basic initial hydrostatic state. The model is

252   integrated for a nondimensional time of $\dfrac{\bar{u}t}{a} = 60$ which corresponds to 8.33 hours without

253   diffusion or viscosity.

254   Fig. 2 shows the numerical and analytic solutions at steady state for the horizontal and

255   vertical velocities, which agree reasonably well. The vertical velocity fields match very

256   closely, although the extrema in the horizontal velocity field are underestimated by the

257   numerical model. The underestimated extrema in the horizontal velocity is also shown in both

258   models of DK83 and Giraldo and Restelli (2008) (hereafter, GR08). And our result in the

259　horizontal velocity is in good agreement with DK83 and GR08.

260　　　Fig. 3 shows the normalized momentum flux values at various times to check vertical

261　transport of horizontal momentum. It is observed that the flux is developing well and the

262　simulations have reached steady-state after $\frac{\bar{u}t}{a} = 60$. It is noted that the mean momentum

263　flux at that time is 97% of its analytic value. It agrees well with DK83 as well as GR08; it is

264　important to point out that the Durran-Klemp model is based on the FD method in both

265　directions while the Giraldo-Restelli model is based on SEM in both directions. The

266　mountain test shows the terrain-following vertical coordinate is well suited for the

267　combination of the horizontal SEM and vertical FDM for spatial discretization even though

268　we consider a small mountain.

269

270　　　b. 2D density current test

271　　　In order to verify the 2DNH's feasibility to control oscillations with numerical viscosity

272　and evaluate numerical schemes in the 2DNH, we conduct the density current test suggested

273　by Straka et al. (1993). The density current test is initialized using a cold bubble in a neutrally

274　stratified atmosphere. When the bubble touches the ground, the density current wave starts to

275　spread symmetrically in the horizontal direction forming Kelvin-Helmholtz rotors. Following

276　Straka et al. (1993) we employ a dynamic viscosity of $\nu = 75$　m^{2}s^{-1} to obtain converged

277　numerical solutions.

278　　　For an initial cold bubble, the potential temperature perturbation is given as

279
$$\theta' = \frac{\theta_c}{2}\left[1 + \cos(\pi r)\right], \tag{24}$$

280　where $\theta_c = -15$　K and $r = \sqrt{\left(\frac{x - x_c}{x_r}\right)^2 + \left(\frac{z - z_c}{z_r}\right)^2}$　with the center of the bubble at

281   $\left(x_c, z_c\right) = \left(0, 3000\right)$ m. No-flux boundary conditions are used for all boundaries. The model

282   is integrated for 900 s on a domain $\left[-25600, 25600\right] \times \left[0, 6400\right]$ m^2.

283   Fig. 4 shows the potential temperature perturbation after 900s for 400, 200, 100, and

284   50m grid spacing ($\Delta\bar{x}$) using 5th order basis polynomials per element. All simulations use

285   $\Delta\bar{z} = 64$ m grid spacing vertically. As expected, the higher resolution experiments produce

286   better solutions than the lower resolution. At the very lowest resolution of 400 m, only two of

287   the three Kelvin-Helmholtz rotors are generated with somewhat coarsened frontal surfaces. In

288   the experiment with 200 m resolution, the three rotors appear but the numerical solution still

289   suffers from coarsening frontal surfaces. The solutions on grids finer than 100 m converge

290   with the three rotor structures adequately simulated. The converged solution is almost

291   identical to other published solutions (e.g. Straka et al. 1993; Skamarock and Klemp 2008;

292   GR08).

293   In order to show the effect of higher order of the basis polynomials, we show the

294   potential temperature perturbations using 8th order basis polynomials per element with the

295   same number of degrees of freedom (DOF) of the simulations using 5th order basis

296   polynomials in Fig. 5. The simulation with 8th order basis polynomials on the very lowest

297   resolution of 400 m reproduced the converged solution more closely than with 5th order basis

298   polynomials. Even in the experiment with 200 m resolution, the coarsening frontal surfaces

299   are mitigated and the solution is similar with the converged solution with three rotors.

300   Fig. 6 shows the profiles of the potential temperature perturbation at the height of 1200

301   m. The results from the highest grid resolution of the simulations using 5th and 8th order

302   basis polynomials are indistinguishable and well converged (Fig. 6a). Three minima

303   corresponding to the three rotors agree well with other published solutions. In addition to the

304   profiles, the front location (-1K of potential temperature perturbation at the surface), and the

14

305 extrema of the pressure perturbation and potential temperature perturbation agree well with

306 each other (Table 1), of which the numbers are comparable to those of GR08. In the

307 numerical results from the different grid resolutions simulated by using 5th order basis

308 polynomials, the potential temperature profile at the coarsest resolution of 400 m grid shows

309 significant fluctuations (Fig. 6b). That of 8th order polynomials, however, tends to be

310 relieved from the deviation from the converged solution (Fig. 6c). The above results suggest

311 that the numerical solution can be converged more rapidly by using higher order of basis

312 polynomial. Furthermore, the results in this paper show that an adequate convergence can be

313 reached at grid resolutions finer than 200 m.

314

315   c. Inertia-gravity wave test

316   This test examines the evolution of a potential temperature perturbation θ' in a

317 constant mean flow with a stratified atmosphere. This initial perturbation diverges to the left

318 and right symmetrically in a channel with periodic lateral boundary conditions. The inertia-

319 gravity wave test introduced by Skamarock and Klemp (1994) (hereafter, SK94) serves as a

320 tool to investigate the accuracy for NH dynamics. Also we use this experiment to check the

321 consistency of the results with various resolutions. The parameters for the test are the same as

322 those of SK94. The initial state is a constant Brunt-Väisälä frequency of $N = 0.01/\text{s}$ with

323 surface potential temperature of $\theta_0 = 300\,\text{K}$ and a uniform zonal wind $\bar{u} = 20\,\text{m/s}$. In order

324 to trigger the wave, the initial potential temperature perturbation θ' is overlaid to the above

325 initial state and is given as

326
$$\theta'\left(x,z\right) = \theta_c \frac{\sin\left(\dfrac{\pi z}{z_c}\right)}{1 + \left(\dfrac{x - x_c}{a_c}\right)} \tag{25}$$

15

327 where $\theta_c = 0.01$ K, $z_c = 10$ km, $x_c = 100$ km. The domain is defined as

328 $(x,z) \in [0,300] \times [0,10]$ km^2. We use periodic lateral boundary conditions and a no-flux

329 boundary conditions for both the bottom and top boundaries. The simulation is performed for

330 3000s with no viscosity.

331 Fig. 7 shows the solution θ' at the initial time and time 3000 s with a horizontal

332 resolution $\Delta \bar{x} = 250$ m and a vertical resolution $\Delta \bar{z} = 250$ m. The figure uses the same

333 contouring interval as in SK94 and Giraldo and Restelli (2008) for comparison. The results

334 are produced with 8th order polynomials per element. We have conducted the 2DNH model

335 with various basis polynomial orders at the same resolution, where the simulated results are

336 found to be very comparable. SK94 give an analytic solution for the case of the Boussinesq

337 equations, but it is only valid for the Boussinesq equations while we use the fully

338 compressible equations in our model. Using the analytic solution only for qualitative

339 comparisons, we find that the extrema of our results are comparable to the analytic values. In

340 comparison with the results of Giraldo and Restelli (2008) in which the fully compressible

341 equations are also used, our results look very similar. Fig. 8 shows the profiles along 5000 m

342 for various horizontal resolutions. All models show consistently identical solutions with the

343 symmetric distribution about the midpoint ($x = 160$ km) which is the location to which the

344 initial perturbation moved by the horizontal flow of 20 m/s after 3000 s. Even at coarser

345 resolution experiments, it does not exhibit phase errors although the maxima and minima near

346 the midpoint ($x = 160$ km) are slightly damped. Table 2 shows the extrema of vertical

347 velocities and potential temperature perturbations for the results of various horizontal

348 resolutions after 3000 s. It is noted that all experiments give almost the same values for

349 potential temperature perturbation where these values in the range

350 $\theta' \in [-1.52 \times 10^{-3}, 2.83 \times 10^{-3}]$ are comparable to other studies (e.g., GR08 and Li et al.

16

351   2013). GR08 give the ranges of $\theta' \in \left[-1.51 \times 10^{-3}, 2.78 \times 10^{-3} \right]$ from the model based on

352   the spectral element and discontinuous Galerkin method. Also Li et al. (2013) show

353    $\theta' \in \left[-1.53 \times 10^{-3}, 2.80 \times 10^{-3} \right]$ using the high-order conservative finite volume model

354   which are similar to our results.

355

356   d. Rising thermal bubble test

357   We also conduct the rising thermal bubble test to verify the consistency of the scheme in

358   the model to simulate thermodynamic motion (Wicker and Skamarock 1998). This test

359   considers the time evolution of warm air in a constant potential temperature environment for

360   an atmosphere at rest atmosphere. The air that is warmer than the ambient air rises due to

361   buoyant forcing which then deforms due to the shearing motion caused by gradients of the

362   velocity field and eventually shapes the thermal bubble into a mushroom cloud. Because the

363   test case has no analytic solution, the simulation results are evaluated qualitatively.

364   The initial conditions we use follow those of GR08 in which the domain for the case is

365   defined as $(x, z) \in \left[0,1 \right]^2$ km^2. We consider no-flux boundary conditions for all four

366   boundaries. The domain is initialized for a neutral atmosphere at rest with $\theta_0 = 300\,\mathrm{K}$ in

367   hydrostatic balance. A potential temperature perturbation to drive the motion is given as

368
$$\theta' = \begin{cases} 0 & \text{for } r > r_c \\ \dfrac{\theta_c}{2}\left[1 + \cos\left(\dfrac{\pi r}{r_c} \right) \right] & \text{for } r \leq r_c \end{cases}, \tag{26}$$

369   where $\theta_c = 0.5\,\mathrm{K}$, $r = \sqrt{\left(x - x_c \right)^2 + \left(z - z_c \right)^2}$ with $\left(x_c, z_c \right) = \left(500, 350 \right)\mathrm{m}$. The model

370   was run for a time of 700 seconds. It should be noted that an explicit second-order diffusion

371   on coordinate surfaces is used with a viscosity coefficient of $\nu = 1$ m^{2}s^{-1} for all simulations

17

372   of this test. The numerical diffusion is applied for momentum and potential temperature along

373   the horizontal and vertical directions so that it eliminates the erroneous oscillations at the

374   small scale – while this amount of diffusion might seem excessive, it has been chosen

375   because it allows the model to remain stable even after the bubble hits the top boundary.

376   Fig. 9 shows the potential temperature perturbation, horizontal wind, and vertical wind

377   fields for the simulations of two resolutions of 20 m and 5 m horizontal and vertical grid

378   spacings ($\Delta \bar{x}$ and $\Delta \bar{z}$), respectively, employing 5th order basis polynomials. In both

379   simulations, the fine structures in the numerical solutions are well depicted with a perfectly

380   symmetric distribution at the midpoint and sharp discontinuities of the fields along boundary

381   lines of the bubble. At lower resolution, however, degradations in the solution are visible in

382   the potential temperature perturbation and vertical wind which are illustrated by fluctuations

383   in the values as well as the concaving contours at the top of the bubble. It is noted that while

384   the numerical solution of the model using the spatially centered FDM of Wicker and

385   Skamarock (1998) shows spurious oscillations in the potential temperature field, the

386   simulations here of 2DNH using the horizontally SEM and vertically FDM is devoid of these

387   oscillations.

388   We also show the vertical profiles of potential perturbation at x = 500 m after 700 s for

389   various resolutions in Fig. 10. Simulations were run with various resolutions of 5, 10, and 20

390   m, where the resolutions given are defined for both the horizontal and vertical directions. The

391   results of 10 m and 5 m resolutions are almost identical to each other. The result of the lowest

392   resolution of 20 m, however, shows a somewhat unresolved solution, in which the maximum

393   value is underestimated and the phase shift is depicted. The time series for maximum

394   potential temperature perturbation and maximum vertical velocity are shown in Fig. 11. In all

395   simulations, the maximum vertical velocity increases as the maximum theta perturbation

396   decreases. This shows that the thermal energy of the theta perturbation leads to the

397 acceleration of the vertical velocity. This result agrees well with the study of Ahmad and

398 Lindeman (2007).

399

6. Summary and Conclusions

401 The non-hydrostatic compressible Euler equations for a dry atmosphere are solved in a

402 simplified 2D slice (X-Z) framework by using the spectral element discretization (SEM) in

403 the horizontal and the third-order finite difference scheme for the vertical discretization. The

404 form of the Euler equations used here are the same as those used in the Weather Research and

405 Forecasting (WRF) model. We employ a hybrid sigma-pressure vertical coordinate which can

406 be modified exactly to the sigma-pressure coordinate at the level of the actual coding

407 implementation.

408 For the spatial discretization, the spatial operators are separated into their horizontal and

409 vertical components. In the horizontal components, the operators are discretized using the

410 SEM in which high-order representations are constructed through the GLL grid points by

411 Lagrange interpolations in the elements. Using GLL points for both interpolation and

412 integration results in a diagonal mass matrix, which means that the inversion of the mass

413 matrix is trivial. In the vertical components, the operators are discretized using the third-order

414 upwind biased finite difference scheme for the vertical fluxes and centered differences for the

415 vertical derivatives. The time discretization relies on the time-split third-order Runge-Kutta

416 technique.

417 We have presented idealized standard benchmark tests for large-scale flows (e.g., linear

418 hydrostatic mountain wave) and for nonhydrostatic-scale flows (e.g., inertia-gravity wave,

419 rising thermal bubble, and density currents). The numerical results show that the present

420 dynamical core is able to produce solutions of good quality comparable to other published

421 solutions. These tests effectively reveal that the combined spatial discretization method of the

422 spectral element and finite difference method in the horizontal and vertical directions,

423 respectively, offers a viable method for the development of a NH dynamical core. Further

424 research will be continued to couple the present core with the existing physics packages

425 together and extend the 2D slice framework to develop a 3D dynamical core for the global

426 atmosphere where the cubed-sphere grid will be used for the spherical geometry.

427

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532

Table Captions

Table 1. Comparison between the 5th and 8th order polynomials per elements for the density current. The simulations is conducted with $\Delta \bar{x} = 50$ m and $\Delta \bar{z} = 50$ m resolution.

Table 2. Comparison of the numerical results for various horizontal resolutions. All simulations use the 8th order polynomials per elements and vertical resolution of $\Delta \bar{z} = 250$ m.

541 Table 1. Comparison between the 5th and 8th order polynomials per elements for the

542 density current. The simulations is conducted with $\Delta \bar{x} = 50$ m and $\Delta \bar{z} = 50$ m resolution.

Order of polynomials	Front location(km)	p'_{max} (Pa)	p'_{min} (Pa)	θ'_{max} (K)	θ'_{min} (K)
5th	14.77	630.62	-452.79	0.08	-8.87
8th	14.74	626.91	-456.84	0.08	-8.94

543

544

545 Table 2. Comparison of the numerical results for various horizontal resolutions for

546 inertia-gravity wave. All simulations use the 8th order polynomials per elements and vertical

547 resolution of $\Delta \bar{z} = 250$ m.

Resolution(m)	w_{max} (m/s)	w_{min} (m/s)	θ'_{max} (K)	θ'_{min} (K)
$\Delta \bar{x} = 125$	2.85×10^{-3}	-2.89×10^{-3}	2.83×10^{-3}	-1.52×10^{-3}
$\Delta \bar{x} = 250$	2.80×10^{-3}	-2.82×10^{-3}	2.83×10^{-3}	-1.52×10^{-3}
$\Delta \bar{x} = 500$	2.73×10^{-3}	-2.73×10^{-3}	2.83×10^{-3}	-1.52×10^{-3}
$\Delta \bar{x} = 750$	2.72×10^{-3}	-2.70×10^{-3}	2.83×10^{-3}	-1.52×10^{-3}
$\Delta \bar{x} = 1250$	2.68×10^{-3}	-2.62×10^{-3}	2.82×10^{-3}	-1.52×10^{-3}

548

549

27

Figure Captions

551   FIG. 1. The grid points of columns within an element having four GLL points. The

552   hybrid sigma coordinate are illustrated and the close (open) circles on the solid (dashed) line

553   indicate the location of the variables at layer mid-points (interfaces).

554

555   FIG. 2. Steady-state flow of (left) horizontal velocity (m/s) and (right) vertical velocity

556   (m/s) over 1 m high mountain at nondimensional time $\dfrac{\bar{u}t}{a} = 60$ with a grid resolution of

557     $\Delta\bar{x} = 2$ km using 5th order basis polynomials per element and $\Delta\bar{z} = 375$ m. The

558   numerical solution is represented by solid lines and the analytic solution by dashed lines.

559

560   FIG. 3. Vertical flux of horizontal momentum, normalized by its analytic value at

561   several non-dimensional times $\dfrac{\bar{u}t}{a}$. Here M and M_H are the momentum flux of the numerical

562   and analytic solution.

563

564   FIG. 4. Potential temperature perturbation after 900 s using (a) $\Delta\bar{x} = 400$ m, (b)

565     $\Delta\bar{x} = 200$ m, (c) $\Delta\bar{x} = 100$ m, and (d) $\Delta\bar{x} = 50$ m grid spacing with 5th order basis

566   polynomials per element. All simulations use $\Delta\bar{z} = 64$ m grid spacing.

567

568   FIG. 5. As in Fig. 4, but with 8th order basis polynomials per element.

569

570   FIG. 6. Profiles of potential temperature perturbation after 900 s along 1200 m height:

571   (a) high-resolution simulations using 5th (thin solid line) and 8th (thick solid line) order basis

572   function, (b) simulations using 5th order basis polynomials, and (c) simulations using 8th

573   order basis polynomials per element. Total number of the degrees of freedom is the same in

574   both 5th and 8th order experiments. All simulations use $\Delta \overline{z} = 64$ m grid spacing.

575

576   FIG. 7. Potential temperature perturbation at the initial time (left) and time 3000s (right)

577   for $\Delta \overline{x} = 250$ m using 8th order basis polynomials per element and $\Delta \overline{z} = 250$ m.

578

579   FIG. 8. The profiles of potential temperature perturbation along 5000 m height for

580    $\Delta \overline{x} = 125$ m (thick solid line), $\Delta \overline{x} = 500$ m (thin dashed line) and $\Delta \overline{x} = 1250$ m (thin

581   solid line) using 8th order basis polynomials per element. All models use $\Delta \overline{z} = 250$ m.

582

583   FIG. 9. Plots of (a,b) potential temperature perturbation (K), (c,d) horizontal wind (m/s),

584   and (e,f) vertical wind (m/s) for the rising thermal bubble test after 700s with (left)

585    $\Delta \overline{x} = 20$ m and (right) $\Delta \overline{x} = 5$ m resolution. All simulations use 5th order basis polynomials

586   per element and $\Delta \overline{z} = 10$ m grid spacing. All negative values are denoted by dashed lines

587   and positive values by solid lines.

588

589   FIG. 10. Vertical profiles of the potential temperature perturbation at x = 500 m after

590   700 s for various resolutions: $\Delta \overline{x}, \Delta \overline{z} = 20$ m (thin solid line), $\Delta \overline{x}, \Delta \overline{z} = 10$ m (thin dashed

591   line), and $\Delta \overline{x}, \Delta \overline{z} = 5$ m (thick solid line).

592

593   FIG. 11. (top) Domain maximum potential temperature perturbation and (bottom)

594   vertical wind for the rising thermal bubble test. All simulations use the 5th order basis

595   polynomials per element, and the vertical resolutions are the same as the horizontal
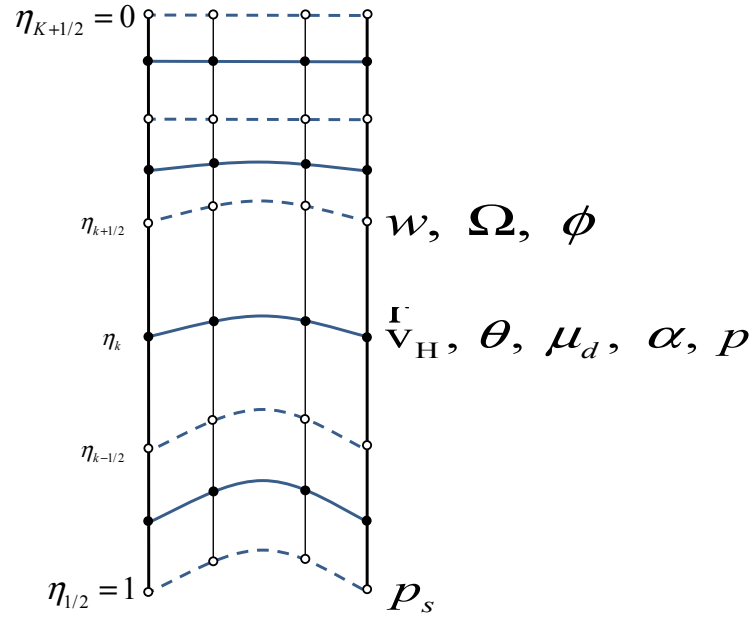
596   resolutions.

597

FIG. 1. The grid points of columns within an element having four GLL points. The hybrid sigma coordinate are illustrated and the close (open) circles on the solid (dashed) line indicate the location of the variables at layer mid-points (interfaces).
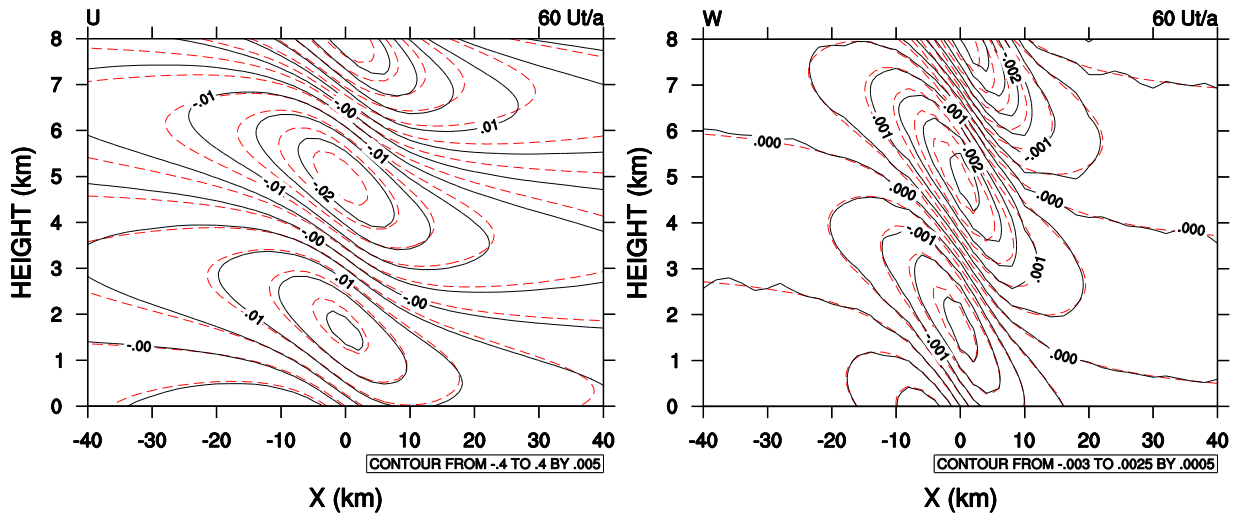
602

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605 $\Delta\bar{x} = 2$ km using 5th order basis polynomials per element and $\Delta\bar{z} = 375$ m. The

606 numerical solution is represented by solid lines and the analytic solution by dashed lines.

607

Vertical flux of horizontal momentum

608

609   FIG. 3. Vertical flux of horizontal momentum, normalized by its analytic value at

610   several non-dimensional times $\dfrac{\bar{u}t}{a}$. Here M and M_H are the momentum flux of the numerical

611   and analytic solutions.

612

613

614   FIG. 4. Potential temperature perturbation after 900 s using (a) $\Delta \bar{x} = 400$ m, (b)

615    $\Delta \bar{x} = 200$ m, (c) $\Delta \bar{x} = 100$ m, and (d) $\Delta \bar{x} = 50$ m grid spacing with 5th order basis

616   polynomials per element for the density current. All simulations use $\Delta \bar{z} = 64$ m grid
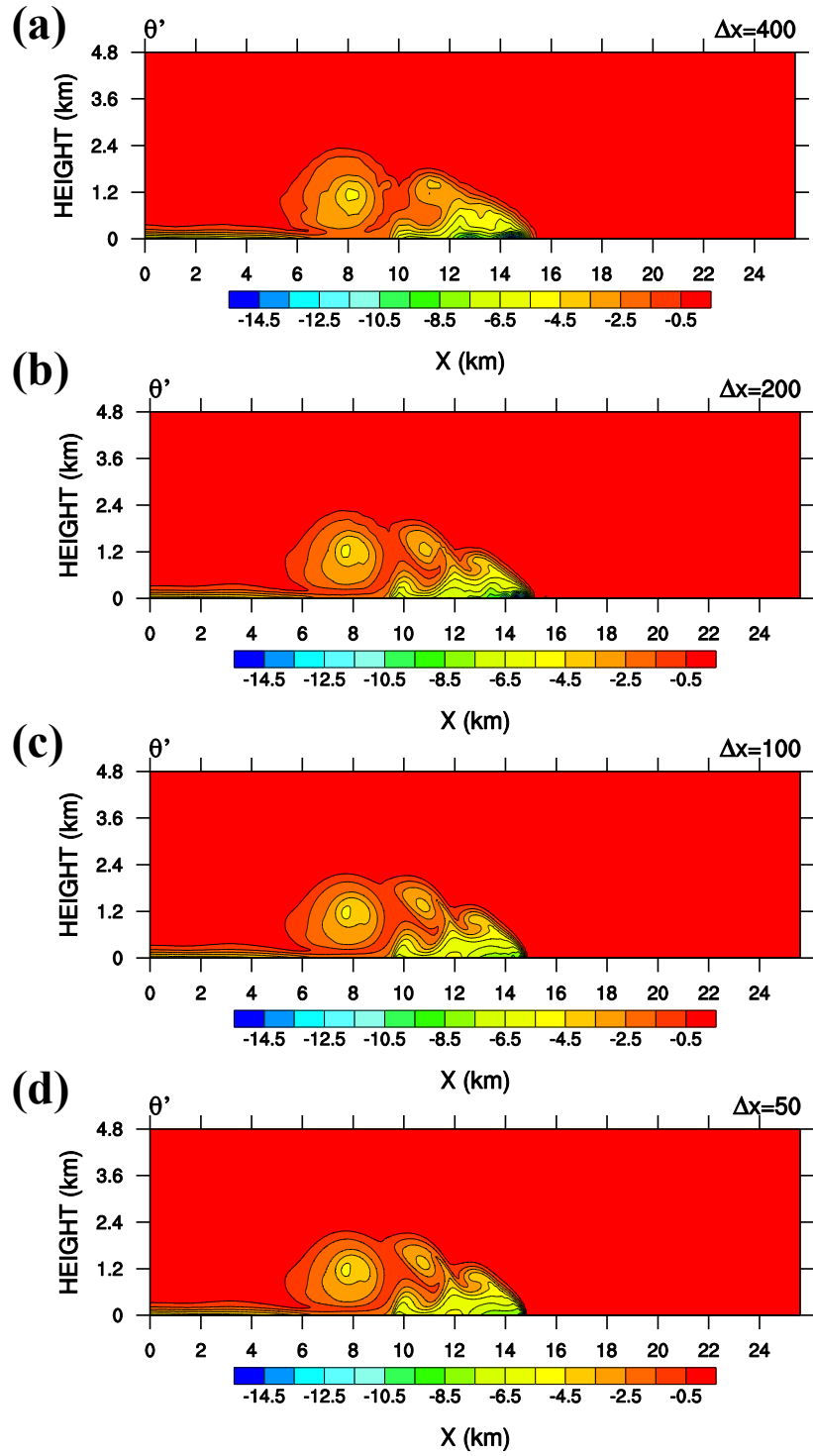
617   spacing.

618

FIG. 5. As in Fig. 4, but with 8th order basis polynomials per element.

622
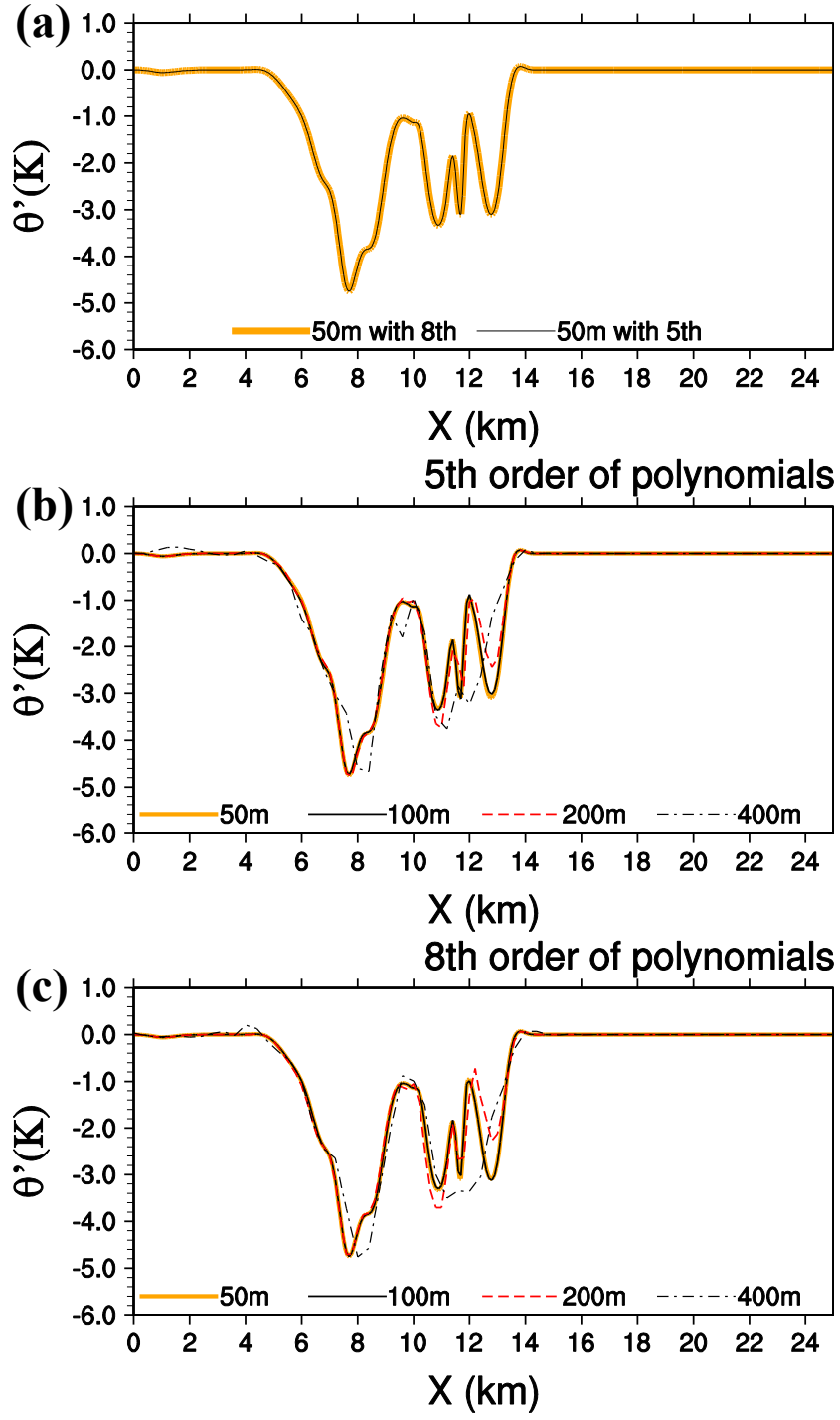
FIG. 6. Profiles of potential temperature perturbation after 900 s along 1200 m height: (a) high-resolution simulations using 5th (thin solid line) and 8th (thick solid line) order basis function, (b) simulations using 5th order basis polynomials, and (c) simulations using 8th order basis polynomials per element. The total number of the degrees of freedom is the same in both 5th and 8th order experiments. All simulations use $\Delta \bar{z} = 64$ m grid spacing.
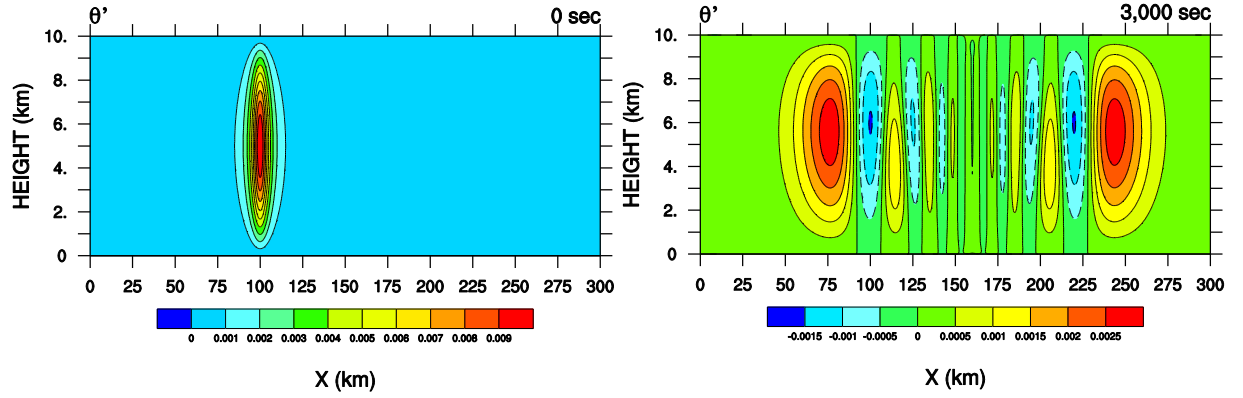
628

629

630   FIG. 7. Potential temperature perturbation at the initial time (left) and time 3000s (right)

631   for $\Delta \bar{x} = 250$ m using 8th order basis polynomials per element and $\Delta \bar{z} = 250$ m for the
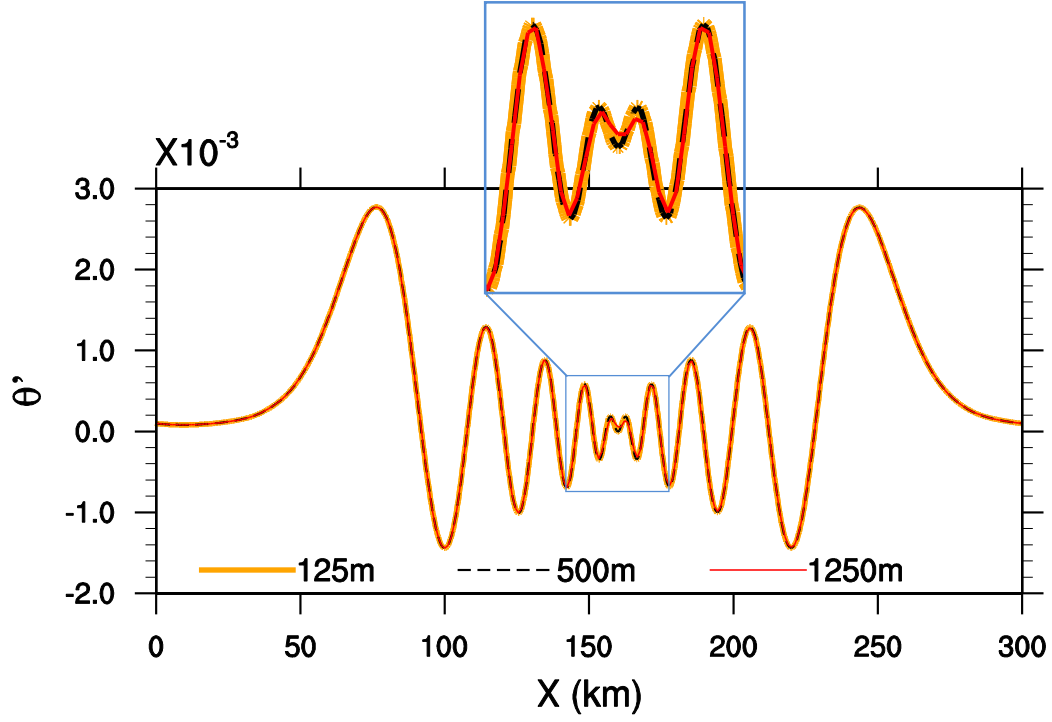
632   inertia-gravity wave.

633

FIG. 8. The profiles of potential temperature perturbation along 5000 m height for $\Delta \bar{x} = 125$ m (thick solid line), $\Delta \bar{x} = 500$ m (thin dashed line) and $\Delta \bar{x} = 1250$ m (thin solid line) using 8th order basis polynomials per element for the inertia-gravity wave. All models use $\Delta \bar{z} = 250$ m.
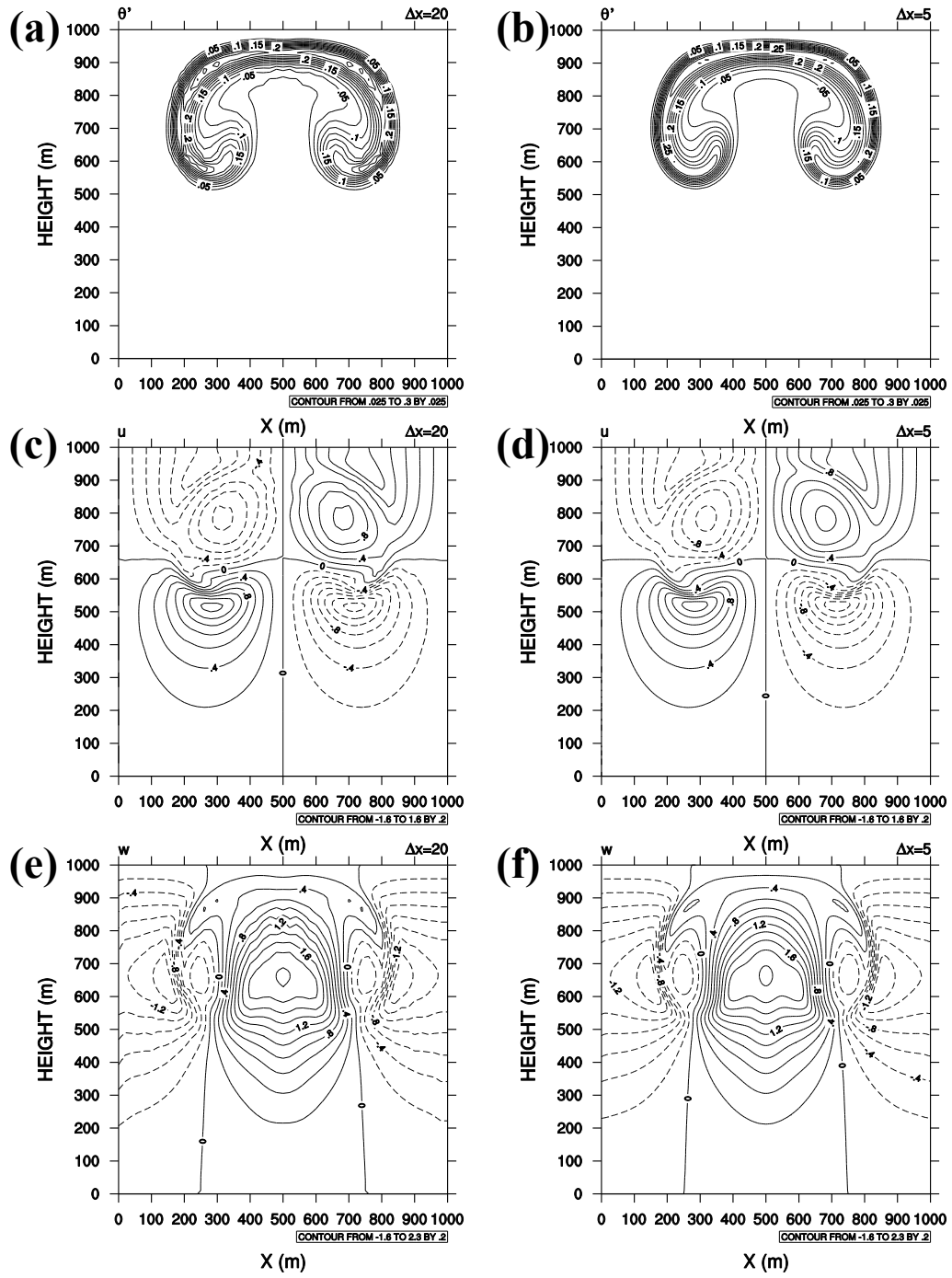
FIG. 9. Plots of (a,b) potential temperature perturbation (K), (c,d) horizontal wind (m/s), and (e,f) vertical wind (m/s) for the rising thermal bubble test after 700s with (left) $\Delta \bar{x}, \Delta \bar{z} = 20\,\mathrm{m}$ and (right) $\Delta \bar{x}, \Delta \bar{z} = 5\,\mathrm{m}$ resolution for the rising thermal bubble test. All simulations use 5th order basis polynomials per element. All negative values are denoted by dashed lines and positive values by solid lines.
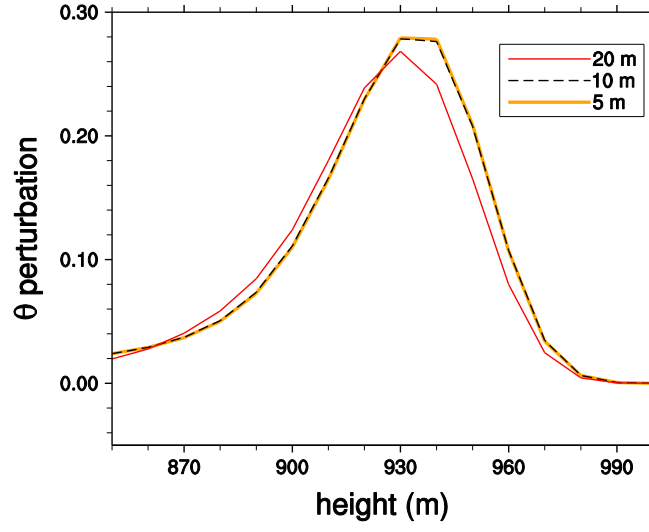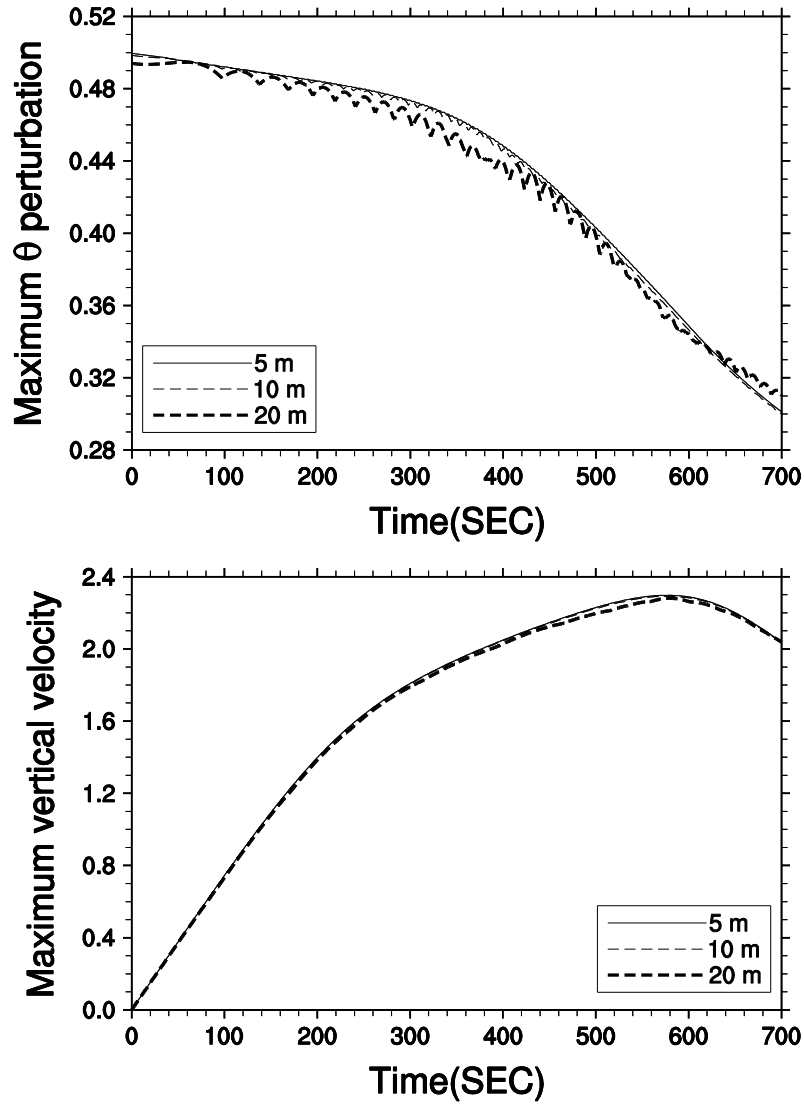
FIG. 10. Vertical profiles of the potential temperature perturbation for the rising thermal bubble test at x = 500 m after 700 s for various resolutions: $\Delta \overline{x}, \Delta \overline{z} = 20$ m (thin solid line), $\Delta \overline{x}, \Delta \overline{z} = 10$ m (thin dashed line), and $\Delta \overline{x}, \Delta \overline{z} = 5$ m (thick solid line).

653

654   FIG. 11. (top) Domain maximum potential temperature perturbation and (bottom)

655 vertical wind for the rising thermal bubble test. All simulations use the 5th order basis

656 polynomials per element, and the vertical resolutions are the same as the horizontal

657 resolutions.